

$1/f^\alpha$ noise and integrable systems

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In this paper, we show that there is a family of well-known integrable systems whose spectral fluctuations decay as $1/f^4$, and thus do not follow the $1/f^2$ law recently conjectured for integrable systems. We present a simple theoretical justification of this fact, and propose an alternative characterization of quantum chaos versus integrability formulated directly in terms of the power spectrum of the spacings of the unfolded spectrum.

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In the absence of a formal definition of quantum chaos, several different tools have been used in the literature to distinguish the chaotic versus integrable character of a quantum system. The most widely used among them is the study of the density of normalized spacings between consecutive levels of the “unfolded” spectrum, which gives a quantitative estimate of the local level fluctuations. More precisely, for quantum systems whose classical limit is chaotic, the celebrated Bohigas-Giannoni-Schmit conjecture [1] posits that this density should coincide with that of a suitable Gaussian ensemble in random matrix theory (RMT). On the other hand, a long-standing conjecture of Berry and Tabor [2] states that the spacings distribution of a “generic” integrable quantum system should follow Poisson’s law. These conjectures have been verified for a large number of systems, both numerically and using semiclassical analytic methods [3].

More recently, an alternative characterization of quantum chaos was proposed by Relaño *et al.* [4]. The basic idea behind this characterization is considering the sequence of energy levels as a time series, whose corresponding power spectrum is then analyzed using standard procedures. It was observed in the latter paper that for all three classical random matrix ensembles the averaged power spectrum of the fluctuations of the spacings exhibits $1/f$ noise, while the noise of a spectrum with a Poissonian spacings distribution behaves as $1/f^2$ (the so-called Brown noise). These numerical observations were theoretically explained using the machinery of RMT in Ref. [5], where it was also conjectured that the $1/f$ ($1/f^2$) spectral noise is in fact a universal property of *all* fully chaotic (integrable) quantum systems. Further numerical simulations are consistent with this conjecture. More precisely, the $1/f$ law has also been detected in the two-body random ensemble [6], whereas a more general power law $1/f^\alpha$ with $1 \leq \alpha \leq 2$ is observed for quantum billiards (or potentials) and random matrix ensembles interpolating between fully chaotic and integrable regimes, with α attaining in these limits the values 1 and 2, respectively [7].

An essential ingredient in the theoretical justification of the $1/f^2$ law for the power spectrum of integrable systems is the Poissonian character of the spacings distribution [5]. However, in a series of recent papers [8–12] it has been shown that there is a wide class of integrable quantum models whose spacings distribution is not Poissonian, namely spin chains of Haldane-Shastry (HS) type [13]. These models, which are the prime examples of integrable spin chains with long-range interactions, appear in connection with sev-

eral phenomena of physical interest such as strongly correlated systems [14], generalized exclusion statistics [15], and the anti de Sitter space-conformal field theory (AdS-CFT) correspondence [16]. It is therefore natural to ascertain whether the $1/f^2$ law conjectured in Ref. [5] applies to spin chains of HS type. These chains are particularly good candidates for a time series analysis, since their spectrum can be exactly computed for a very large number of sites, and there is ample numerical and theoretical evidence that their level density (whose knowledge is essential for unfolding the spectrum) becomes Gaussian when the number of sites is sufficiently large [8–12,17,18]. In this paper we shall focus on the simplest type of HS chains, whose raw spectrum is equally spaced (as in the Polychronakos chain) or almost equally spaced (as in the original Haldane-Shastry chain). Our main result is that the power spectrum of these chains—unlike all the integrable models previously studied in the literature—clearly obeys the $1/f^4$ power law, characteristic of black noise. In order to highlight the main ideas and simplify the theoretical derivation we shall primarily concentrate on the case of an equispaced raw spectrum, giving only a qualitative justification of our result in the general case.

Let $E_1 < \dots < E_{n+1}$ be the energies of a quantum system, with (in general distinct) degeneracies D_i , and denote by μ and σ the mean and standard deviation of the spectrum. In accordance with the previous remark, we shall assume that the following conditions are satisfied:

(i) The energies are equispaced, i.e., $E_{j+1} = E_1 + jd$.

(ii) The continuous part of the level density is a Gaussian with parameters μ and σ , with $\sigma \gg \sqrt{nd}$.

The requirement that $\sigma \gg \sqrt{nd}$, which shall be needed in what follows, is satisfied by all spin chains of HS type studied so far when the number of sites is sufficiently large [10–12]. By the second condition, the unfolded spectrum is given by $\epsilon_i = \frac{1}{2} [1 + \text{erf}(\frac{E_i - \mu}{\sqrt{2}\sigma})]$. The i th normalized spacing of the unfolded spectrum is then defined by $s_i = n(\epsilon_{i+1} - \epsilon_i) / (\epsilon_{n+1} - \epsilon_1)$, so that the mean spacing is one. Following [4], we shall characterize the fluctuations of the spacings from their mean by the statistic $\delta_l = \sum_{j=1}^l (s_j - 1)$, $l = 1, \dots, n$, whose discrete Fourier transform is given by

$$\hat{\delta}_k = \frac{1}{\sqrt{n}} \sum_{l=1}^n \delta_l e^{-2\pi i k l / n}, \quad k = 1, \dots, n. \quad (1)$$

In this paper we shall be interested in the behavior of the power spectrum of the statistic δ_l , defined by $P(k) = |\hat{\delta}_k|^2$,

$k=1, \dots, n$. Note that, since the δ_j 's are real, $\hat{\delta}_{n-k}$ is the complex conjugate of $\hat{\delta}_k$, so that $P(n-k)=P(k)$. For this reason, from now on we shall take k in the range $1, 2, \dots, [n/2]$.

To begin with, we shall prove a remarkable identity expressing $P(k)$ directly in terms of the Fourier transform of the spacings s_j . Indeed, taking into account that $\sum_{l=1}^n l e^{-2\pi i k l / n} = \frac{in}{2} e^{-k\pi i / n} / \sin(\frac{k\pi}{n})$, from Eq. (1) we have

$$\hat{\delta}_k = \frac{1}{\sqrt{n}} \sum_{l=1}^n \sum_{j=1}^l s_j e^{-2\pi i k l j / n} + \frac{\sqrt{n}}{2i} \frac{e^{-k\pi i / n}}{\sin\left(\frac{k\pi}{n}\right)}. \quad (2)$$

The first term in the previous formula can be expressed as $n^{-1/2} \sum_{j=1}^n f(j) s_j$, with

$$f(j) = \sum_{l=j}^n e^{-2\pi i k l j / n} = \frac{e^{k\pi i j / n}}{2i \sin\left(\frac{k\pi}{n}\right)} (e^{-2\pi i k j / n} - e^{-2\pi i k l / n}).$$

Since $\sum_{j=1}^n s_j = n$, the previous equality and Eq. (2) yield the exact identity

$$P(k) = \frac{|\hat{s}_k|^2}{4 \sin^2\left(\frac{k\pi}{n}\right)}. \quad (3)$$

It should be emphasized that Eq. (3) is in fact valid for *any* finite spectrum. The latter equation shall be the starting point to derive an analytic approximation to the power spectrum of an energy spectrum satisfying conditions (i) and (ii) above.

In practice, to smooth out spurious fluctuations one usually divides the whole spectrum into several subspectra with an equal number of levels, and determines the average $\langle P(k) \rangle$ of the individual power spectra. For this reason, we shall consider a subspectrum with energies $E_j = E_1 + (j-1)d$, where $j = l_0, \dots, l_1 \equiv l_0 + m$ and $m \gg 1$. Calling $\nu = m / (\epsilon_1 - \epsilon_0)$, the spacings $s_i = \nu(\epsilon_{l_0+i} - \epsilon_{l_0+i-1})$, $i = 1, \dots, m$, are approximately given by

$$s_i \approx \nu F'(E_{l_0+i-1}) d = \frac{\nu d}{\sqrt{2\pi\sigma}} e^{-(E_{l_0+i-1} - \mu)^2 / (2\sigma^2)}.$$

The discrete Fourier transform of these spacings is then

$$\hat{s}_k \approx \frac{\nu d}{\sqrt{2\pi m \sigma}} \sum_{l=1}^m e^{-(E_{l_0} - \mu + (l-1)d)^2 / (2\sigma^2)} e^{-2\pi i k l / m}.$$

When $k \ll m$, the previous sum can be approximated with great accuracy by an integral, so that

$$\hat{s}_k \approx \frac{\nu}{\sqrt{2\pi m \sigma}} e^{2\pi i k (E_{l_0} - \mu - d) / (md)} \int_{E_{l_0} - \mu}^{E_{l_1} - \mu} e^{-x^2 / (2\sigma^2)} e^{-2\pi i k x / (md)} dx.$$

Setting $x_j = (E_j - \mu) / (\sqrt{2}\sigma)$, $y = \sqrt{2}\pi\sigma k / (md)$, $z_j = x_j + iy$, with $j=0, 1$, and using $\{[19], \text{Eq. (7.4.32)}\}$ and Eq. (3), we finally obtain

$$P(k) \approx \frac{m e^{-2y^2}}{4 \sin^2\left(\frac{\pi k}{m}\right)} \left| \frac{\text{erf } z_0 - \text{erf } z_1}{\text{erf } x_0 - \text{erf } x_1} \right|^2. \quad (4)$$

If one is dealing with the whole spectrum, then $m=n$, $\text{erf } x_1 \approx 1$, $\text{erf } x_0 \approx -1$ and Eq. (4) simplifies to

$$P(k) \approx \frac{n e^{-2y^2}}{16 \sin^2\left(\frac{\pi k}{n}\right)} |\text{erf } z_0 - \text{erf } z_1|^2. \quad (5)$$

We shall next use Eq. (4) to determine the asymptotic behavior of $P(k)$ in the range $k_0 \ll k \ll m$, where $k_0 \equiv (md/\sigma) \max(1, |x_0|, |x_1|) \ll m$ by the condition $\sigma \gg \sqrt{nd}$. Since $k \ll m$, we can write Eq. (4) as

$$P(k) \approx \frac{m^3 \varphi(k)}{4\pi^2 k^2 (\text{erf } x_0 - \text{erf } x_1)^2}, \quad (6)$$

where $\varphi(k) \equiv e^{-2y^2} |\text{erf } z_0 - \text{erf } z_1|^2$. The condition $k \gg k_0$ implies that

$$|z_j| > y \gg 1, \quad y \gg |x_j|, \quad j=0, 1, \quad (7)$$

so that the well-known asymptotic formula $\{[19], \text{Eq. (7.1.23)}\}$ $1 - \text{erf } z \approx e^{-z^2} / (\sqrt{\pi} z)$, valid for $|z| \gg 1$ and $|\arg z| < 3\pi/4$, holds for both $z=z_0$ and $z=z_1$. Taking into account that $2y(x_1 - x_0) = 2\pi k \in \mathbb{Z}$ we easily obtain

$$\varphi(k) \approx \frac{1}{\pi} \left| \frac{e^{-x_0^2}}{z_0} - \frac{e^{-x_1^2}}{z_1} \right|^2. \quad (8)$$

If $x_0^2 \approx x_1^2$, which is only possible when $x_1 \approx -x_0 > 0$, i.e., when the subspectrum under consideration is approximately symmetric with respect to μ , Eq. (8) immediately yields $\varphi(k) \approx e^{-2x_1^2} (x_0 - x_1)^2 / (\pi |z_0 z_1|^2) \approx 4x_1^2 e^{-2x_1^2} / (\pi y^4)$. Substituting into Eq. (6) we finally obtain the asymptotic power law

$$P(k) \approx \frac{m^7 d^4 x_1^2 e^{-2x_1^2}}{16\pi^7 \sigma^4 (\text{erf } x_1)^2 k^6}. \quad (9)$$

In particular, if we are dealing with the full spectrum we have the slightly simpler relation

$$P(k) \approx \frac{n^7 d^4 x_1^2 e^{-2x_1^2}}{16\pi^7 \sigma^4 k^6}. \quad (10)$$

In the generic case, i.e., when $x_0^2 \neq x_1^2$, the leading term of the expansion of Eq. (8) in powers of $1/y$ becomes $\varphi(k) \approx (e^{-x_0^2} - e^{-x_1^2})^2 / (\pi y^2)$. Substituting again into Eq. (6) we obtain

$$P(k) \approx \frac{m^5 d^2 (e^{-x_0^2} - e^{-x_1^2})^2}{8\pi^5 \sigma^2 (\text{erf } x_0 - \text{erf } x_1)^2 k^4}. \quad (11)$$

Equation (11) is the key analytic result for establishing the $1/f^4$ behavior of the averaged power spectrum $\langle P(k) \rangle$ for spin chains of HS type. Indeed, when one averages the individual $P(k)$ of a large number of subspectra, at most one of these subspectra can be symmetric about μ . Hence, if the total number of subspectra is sufficiently large we must have

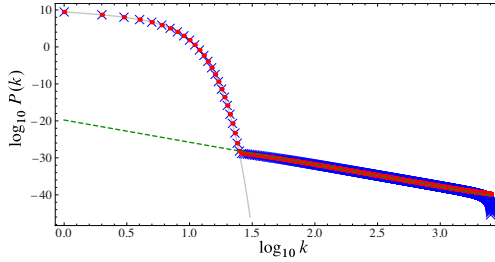


FIG. 1. (Color online) log-log plot of $P(k)$ for the whole spectrum of the spin chain (13) with 100 sites (blue crosses), compared to its analytic approximation (5) (red dots), the power law (10) (dashed green line), and the approximation (16) (gray curve).

$$\langle P(k) \rangle = \frac{m^5 d^2}{8\pi^5 \sigma^2} \left\langle \left(\frac{e^{-x_0^2} - e^{-x_1^2}}{\operatorname{erf} x_0 - \operatorname{erf} x_1} \right)^2 \right\rangle \frac{1}{k^4}, \quad (12)$$

where m is the number of spacings in each subspectrum.

In order to test our theoretical results, we shall first consider the ferromagnetic rational chain of type B and spin $\frac{1}{2}$ introduced in [20], whose Hamiltonian is given by

$$\mathcal{H}^\epsilon = \sum_{1 \leq i \neq j \leq N} \left[\frac{1 - S_{ij}}{(\xi_i - \xi_j)^2} + \frac{1 - S_i S_j S_{ij}}{(\xi_i + \xi_j)^2} \right] + \beta \sum_{i=1}^N \frac{1 - \epsilon S_i}{\xi_i^2}. \quad (13)$$

Here $\beta > 0$, $\epsilon = \pm 1$, S_{ij} is the operator which permutes the i th and j th spins, S_i is the operator reversing the i th spin, and $\xi_i = \sqrt{2}t_i$, where t_i is the i th zero of the generalized Laguerre polynomial $L_N^{\beta-1}$. It has recently been shown [10] that the partition function of this model is given by $Z(q) = \prod_{j=1}^N (1+q^j)$, where $q \equiv e^{-1/(k_B T)}$, so that condition (i) above is satisfied with

$$E_1 = 0, \quad d = 1, \quad n = \frac{1}{2}N(N+1). \quad (14)$$

It was rigorously proved in Ref. [18] that when the number of sites N tends to infinity the level density tends to a Gaussian with parameters

$$\mu = \frac{N}{4}(N+1), \quad \sigma^2 = \frac{N}{12} \left(N + \frac{1}{2} \right) (N+1). \quad (15)$$

The spectrum is obviously symmetric about its mean, so that if we consider the whole spectrum $P(k)$ should approximately be given by Eq. (5) for all $k \ll N^2$, and follow the power law (10) when $N \ll k \ll N^2$. We have verified that this prediction is in excellent agreement with the numerical data for a wide range of values of N , up to $N=100$. For instance, in the latter case it is apparent from Fig. 1 that the approximation (10) is extremely accurate even for k close to $[n/2]$. As to the $1/f^6$ power law, the relative root mean square (rms) error in the fit of the \log_{10} of the RHS of Eq. (10) to $\log_{10} P(k)$ in the range $50 \leq k \leq 1000 \approx n/5$ is only 6.9×10^{-4} . The behavior of $P(k)$ for small k seen in Fig. 1 perhaps deserves a brief explanation. To this end, note that when $k \ll (nd/\sigma) \min(|x_0|, |x_1|)$, which in this case is tantamount to $k \ll k_0 = O(N)$, instead of Eq. (7) we have $y \ll \min(|x_0|, |x_1|)$.

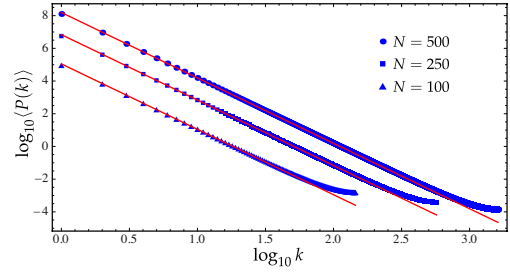


FIG. 2. (Color online) log-log plot of $\langle P(k) \rangle$ for the spin chain (13) (blue marks), compared to its analytic approximation (12) (red lines). Note that the initial (curved) part of the plot of $P(k)$ observed in Fig. 1 is absent in this case, since $(md/\sigma) \min(|x_0|, |x_1|) \leq 1$ for all subspectra involved.

Since, by Eqs. (14) and (15), $|x_0| = -x_0$ and $|x_1| = x_1$ are $O(N^{1/2})$, when N is large enough both $-z_0$ and z_1 are close to the positive real axis and their modules are at least $O(N^{1/2})$, so that $\operatorname{erf}(-z_0)$, $\operatorname{erf} z_1 \approx 1$. From Eq. (5) we then obtain

$$P(k) \approx \frac{ne^{-2y^2}}{4 \sin^2\left(\frac{\pi k}{n}\right)}, \quad k \ll k_0. \quad (16)$$

This is indeed an excellent approximation to $P(k)$ for small k , as can be seen from Fig. 1.

Even though the spectrum of the chain (13) is symmetric about μ , the $1/f^4$ power law predicted for spin chains of HS type clearly emerges when one considers the averaged power spectrum $\langle P(k) \rangle$ of even a relatively small number of subspectra. To be more precise, we have divided the energy range $[\mu - 5\sigma, \mu + 5\sigma]$ into ten subspectra of equal length $m = [\sigma]$ and computed numerically the resulting $\langle P(k) \rangle$ for a wide range of values of N , up to $N=500$. As can be seen in Fig. 2, Eq. (12) is in excellent agreement with the numerical data except for k close to $[m/2]$ in the logarithmic scale. The accuracy of this approximation steadily improves as N increases; for instance, the relative rms error of the approximation (12) in a log-log plot for $10 \leq k \leq [m/6]$ and $N=100, 250$, and 500 is, respectively, given by 0.065, 0.046, and 0.033.

Consider next the spin $\frac{1}{2}$ (antiferromagnetic) Haldane-Shastry chain, with Hamiltonian

$$\mathcal{H} = \frac{1}{2} \sum_{1 \leq i < j \leq N} \frac{1 + S_{ij}}{\sin^2(\theta_i - \theta_j)}, \quad \theta_j = \frac{j\pi}{N}. \quad (17)$$

Although the partition function of this chain has been evaluated in closed form [8], in practice its complicated structure precludes the calculation of the spectrum for $N \geq 40$. On the other hand, the mean and variance of the energy can be exactly computed for all N by taking suitable traces in Eq. (17), with the result [8] $\mu = \frac{N}{8}(N^2 - 1)$ and $\sigma^2 = \frac{N}{480}(N^2 - 1)(N^2 + 11)$. It was also numerically shown in the latter reference that for sufficiently large N the level density is well approximated by a normal distribution with the above parameters. As is well known, the spectrum of the Haldane-Shastry chain (17) is not equispaced. However, for large N condition (i) above is approximately satisfied, since the vast majority of the differ-

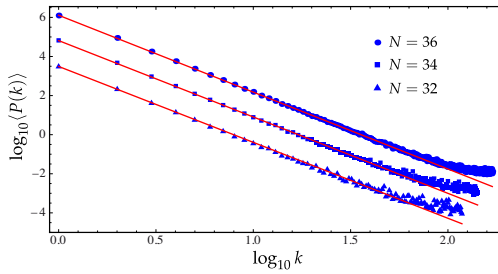


FIG. 3. (Color online) Plot of $\log_{10}\langle P(k) \rangle$ for the spin chain (17) (blue marks) fitted by a straight line (red) in the range $0 \leq \log_{10} k \leq \log_{10}[m/6]$. The plots for $N=32$ and $N=36$ have been displaced to avoid overlapping.

ences $d_i \equiv E_{i+1} - E_i$ are equal to $d=1$ (for even N) or $d=2$ (for odd N), and the differences $d_i \neq d$ actually correspond to energies E_i in the tail of the Gaussian level density; cf. Ref. [11]. As to condition (ii), the requirement $\sigma \gg \sqrt{nd}$ is also fulfilled for large N , since $\sigma = O(N^{5/2})$, while it was shown in [11] that $n \leq E_{n+1} - E_1 = O(N^3)$. For these reasons, it is to be expected that the power spectrum of the Haldane-Shastry chain obey the $1/f^4$ power law when the number of sites is sufficiently large. We have numerically verified that this law clearly holds when N ranges from 26 to 36. For instance, in Fig. 3 we present the results for $N=32, 34, 36$, where $\langle P(k) \rangle$ was computed by averaging over ten subspectra of equal maximal length m in the interval $[\mu - 5\sigma, \mu + 5\sigma]$. A least-squares fit of $\log_{10}\langle P(k) \rangle$ to $\beta - \alpha \log_{10} k$ in the range $1 \leq k \leq [m/6]$ yields an optimum α of 3.884, 3.908, and 3.933 for $N=32, 34$, and 36 , respectively. The corresponding values of the squared correlation coefficient r^2 are 0.9975, 0.9989, and 0.9996, which strongly suggests that the $1/f^4$ power law also

holds in this case when N is sufficiently large.

In conclusion, the results of this paper show that there is a whole family of integrable systems, namely, spin chains of Haldane–Shastry type, whose spectral fluctuations clearly exhibit $1/f^4$ black noise rather than the $1/f^2$ noise conjectured in Ref. [5]. Note, however, that our findings do not invalidate the theoretical justification for the $1/f^2$ law proposed in the latter reference. Indeed, an essential assumption of this justification is that the spacings distribution be Poissonian, which is certainly not the case for spin chains of HS type [8–12]. Given the fact that the spectrum of a chaotic system features $1/f^\alpha$ noise with $1 \leq \alpha < 2$, and that spin chains of HS type possess a higher degree of integrability than generic integrable systems due to their underlying Yangian symmetry, it is tempting to conclude that the exponent α in the $1/f^\alpha$ spectral noise provides a quantitative measure of the degree of integrability of a system. It would be of considerable interest in this respect to find integrable quantum systems featuring $1/f^\alpha$ noise with $2 < \alpha \leq 4$. Finally, a noteworthy by-product of our analysis is the universal identity (3), which shows that $P(k) \propto |\hat{s}_k|^2/k^2$ for $k \ll n$. This fact makes it possible to translate any statement about the behavior of the power spectrum of the statistic δ_i into a simpler statement on the power spectrum of the spacings s_i . From this alternative point of view, generic integrable systems are characterized by white noise in the fluctuations of their spacings s_i , while in all chaotic systems the corresponding noise actually grows with the frequency.

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